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A RELATION FOR DOMINO ROBINSON-SCHENSTED ALGORITHMS

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Abstract. We describe a map relating hyperoctahedral Robinson-Schensted algorithms on standard domino tableaux of unequal rank. Iteration of this map relates the algorithms defined by Garfinkle and Stanton–White and when restricted to involutions, this construction answers a question posed by M. A. A. van Leeuwen. The principal technique is derived from operations defined on standard domino tableaux by D. Garfinkle which must be extended to this more general setting.

1. Introduction

The classical Robinson-Schensted algorithm defines a bijection between the elements of the symmetric group S_n and same-shape pairs of standard Young tableaux of size n. The work of Garfinkle [3] defines similar bijections for H_n, the hyperoctahedral group on n letters, using pairs of certain same-shape standard *domino* tableaux as parameter sets.

Viewing H_n as the Weyl group of a simple Lie group of type C, Garfinkle's generalization is a map G_0 whose image is precisely the set of same-shape pairs of standard domino tableaux of size n and rank 0. When viewing H_n as the Weyl group of a simple Lie group of type B, she defines a more natural map G_1 whose image is the set of same-shape pairs of standard domino tableaux of size n and rank one. M. A. A. van Leeuwen has observed that Garfinkle's definition can be extended to define bijective maps G_r from H_n to same-shape pairs of standard domino tableaux of arbitrary rank r [11]. For r sufficiently large, G_r recovers the bijection of Stanton and White defined between H_n and pairs of same-shape standard bitableaux (cf. [10] and also [8]).

Consider an element $\sigma \in H_n$ and let $(T, S) = G_r(\sigma)$ and $(T', S') = G_{r+1}(\sigma)$. The main result of this paper describes a map between the pairs (T, S) and (T', S') using techniques from [3]. In this way, we obtain maps that relate the different members of this family of generalized Robinson-Schensted algorithms as well as the algorithm and Stanton and White. When σ is an involution, the map sending (T, S) to (T', S') has a particularly simple description and answers a question posed by M. A. A. van Leeuwen in [11], p. 26.

The combinatorial results of this paper are particularly relevant to recent results in the study of the Kazhdan-Lusztig cell structure of an unequal parameter Hecke algebra $\mathcal{H}$. Garfinkle's original work on the primitive spectrum of a universal enveloping algebra of a complex semisimple Lie algebra classified the Kazhdan-Lusztig

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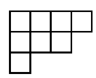
cell structure of equal parameter Hecke algebras of type B_n. In the more general setting of unequal parameter $\mathcal{H}$, [1] conjectures a parametrization of cells via domino tableaux of rank r, where the specific choice of r depends on the underlying parameters of $\mathcal{H}$. In [9], the results of the present paper are used to reconcile the above conjecture and Garfinkle's original work on primitive ideals. In related work, [6] and [5] provide a geometric interpretation of these combinatorial results in the setting of rational Cherednik algebras.

2. Definitions and Preliminaries

2.1. **Generalized Robinson-Schensted Algorithms.** Following Garfinkle [3], we view the elements of the hyperoctahedral group H_n as those subsets σ of $\mathbb{N}_n \times \mathbb{N}_n \times \{\pm 1\}$, with $\mathbb{N}_n = \{1, 2, \ldots, n\}$, such that the projections onto the first and second components of σ are always bijections onto $\mathbb{N}_n$ ([3], (1.1.2)). We will write the element σ as $\{(\sigma_1, 1, \epsilon_1), \ldots, (\sigma_n, n, \epsilon_n)\}$. In this form, σ corresponds to the signed permutation $(\epsilon_1\sigma_1, \epsilon_2\sigma_2, \ldots, \epsilon_n\sigma_n)$.

For us, Young diagrams will be finite left-justified arrays of squares arranged with non-increasing row lengths. A square in row i and column j of the diagram will be denoted $S_{i,j}$ so that $S_{1,1}$ is the uppermost left square in the Young diagram below:



Definition 2.1. Let $r \in \mathbb{N}$ and λ be a partition of a positive integer m. A *domino tableau of rank r and shape λ* is a Young diagram of shape λ whose squares are labeled by integers from some set M in such a way that 0 labels the square (i, j) iff $i + j < r + 2$, each element of M labels exactly two adjacent squares, and all labels increase weakly along both rows and columns. A domino tableau is *standard* iff $M = \mathbb{N}_n$ for some n.

We will write $DT_r(\lambda)$ for the family of all domino tableaux of rank r and shape λ and $DT_r(n)$ for the family of all domino tableaux of rank r which contain exactly n dominos. The corresponding families of standard tableaux will be denoted $SDT_r(\lambda)$ and $SDT_r(n)$. The set of squares in a tableau T labeled by the integer l will be denoted by $supp(l, T)$ and $supp(0, T)$ will be called the *core* of T.

Following [3] and [11], we describe the Robinson-Schensted bijections

$$G_r : H_n \rightarrow SDT_r(n) \times SDT_r(n).$$

The algorithm is based on an insertion map α which, given an element (i, j, ϵ) of $\sigma \in H_n$, inserts a domino with label i into a domino tableau.

Definition 2.2. Consider $\sigma \in H_n$, $(i, j, \epsilon) \in \sigma$, and a domino tableau $T' \in DT_r(k)$. Write $\ell = \{l_1, l_2, \ldots, l_k\}$ for the set of labels of the dominos of T' listed in increasing order. When $i \notin \ell$, we can define a tableau $T = \alpha((i, j, \epsilon), T') \in DT_r(k + 1)$ by the following procedure:

(1) If $i > l_k$, T is formed by:
   (a) adding a new horizontal domino with label i to the end of the first row of T' if $\epsilon = 1$, or by
   (b) adding a new vertical domino with label i at the end of the first column of T' if $\epsilon = -1$.

(2) Otherwise, let l_m be the least label in ℓ greater than i. We inductively define a sequence $\{T_{m-1}, T_m, \ldots, T_{k+1}\}$ of domino tableaux and let $T = T_{k+1}$. To this effect, construct T_{m-1} by removing all dominos with labels greater or equal to l_m from T'. Let $T_m = \alpha((i,j,\epsilon), T_{m-1})$. For $p \geq m$,
   (a) if $supp(l_p, T') \cap T_p = \emptyset$, then T_{p+1} is the tableau obtained from T_p by labeling $supp(l_p, T')$ with the integer l_p;
   (b) if $supp(l_p, T') \cap T_p = \{S_{ij}\}$, then T_{p+1} is the tableau obtained from T_p by labeling $\{S_{i,j+1}, S_{i+1,j+1}\}$ with the integer l_p if $supp(l_p, T')$ is horizontal, or by labeling $\{S_{i+1,j}, S_{i+1,j+1}\}$ with the integer l_p if $supp(l_p, T')$ is vertical.
   (c) if $supp(l_p, T') \cap T_p = supp(l_p, T')$, then T_{p+1} is the tableau obtained by adding a horizontal domino with label l_p at the end of row $\iota + 1$ of T_p if $supp(l_p, T')$ is horizontal and lies in row ι of T', or by adding a vertical domino with label l_p at the end of column $\iota + 1$ of T_p if $supp(l_p, T')$ is vertical and lies in column ι of T'.

That this procedure is well-defined and indeed produces a domino tableau is verified in [3], Section 2. To describe the generalized Robinson-Schensted algorithm itself, we start by constructing the left tableau. Let $T(0)$ be the only tableaux in $SDT_r(0)$. Define $T(1) = \alpha((\sigma_1, 1, \epsilon_1), T(0))$ and continue inductively by letting

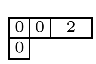
$$T(k+1) = \alpha\big((\sigma_{k+1}, k+1, \epsilon_{k+1}), T(k)\big).$$

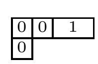
The left domino tableau $T(n)$ will be standard and of rank r. The right tableaux track the shapes of the left tableaux. Begin by forming a domino tableau $S(1)$ by adding a domino with label 1 to $T(0)$ in such a way that $S(1)$ and $T(1)$ have the same shape. Continue adding dominos by requiring that at each step $S(k)$ lie in $SDT_r(k)$ and have the same shape as $T(k)$. Again, the domino tableau $S(n)$ will be standard and of rank r. Finally, the image of σ under G_r is defined as the tableau pair $(T(n), S(n))$. To simplify notation, we will write $G_r^k(\sigma)$ for the pair $(T(k), S(k))$. We will also sometimes simplify notation slightly and write $\alpha_m(T)$ instead of $\alpha\big((\sigma_m, m, \epsilon_m), T\big)$ and $\alpha_m((T,S))$ for the domino tableau pair obtained by following the above shape-tracking procedure for $\alpha_m(T)$.

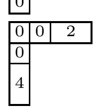
When $r = 0$ or 1, the G_r are precisely Garfinkle's algorithms; for $r > 1$ they are natural extensions to larger-rank tableaux. In all cases, G_r defines a bijection from H_n to pairs of same-shape tableaux in $SDT_r(n)$ [11]. These generalizations of the Robinson-Schensted algorithm share a number of properties with the original algorithm. We state the following:

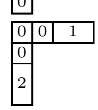
Proposition 2.3. ([11], (4.2)) *$G_r(\sigma^{-1}) = (S,T)$ whenever $G_r(\sigma) = (T,S)$. In particular, if σ is an involution, $G_r(\sigma) = (T,T)$ for some standard domino tableau T.*

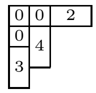
Example 2.4. Consider the signed permutation $(2\ -4\ -3\ 1)$. It corresponds to the set $\sigma = \{(2,1,1), (4,2,-1), (3,3,-1), (1,4,1)\} \in H_4$. If $r = 2$, then successive insertion of elements of σ into the empty tableau of rank zero yields the following sequence of tableau pairs

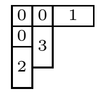
$T(1) =$ 

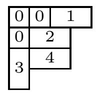
$S(1) =$ 

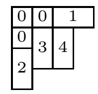
$T(2) =$ 

$S(2) =$ 

$T(3) =$ 

$S(3) =$ 

$T(4) =$ 

$S(4) =$ 

Consequently, $G_2(\sigma) = (T(4), S(4))$.

2.2. **Cycles.** The notion of a cycle in a domino tableau appears in a number of references. See for instance [2], [12], or [13]. We now review its definition.

Definition 2.5. For a standard domino tableau T of arbitrary rank r, we will call a square in position (i, j) *fixed* when $i + j$ has the opposite parity as r, otherwise, we will call it *variable*.

It is possible to choose the sets of fixed and variable squares differently, as in [3],(1.5.4); however, we refrain from defining the more general possibilities as only this choice will be necessary for our results.

If $T \in SDT_r(n)$, we will write $D(k, T)$ for the domino labeled by the positive integer k in T viewed as a set of labeled squares, and $supp\, D(k, T)$ will denote its underlying squares. Write $label\, S_{i,j}$ for the label of the square $S_{i,j}$ in T. We extend this notion slightly by letting $label\, S_{i,j} = 0$ if either i or j is less than or equal to zero, and $label\, S_{i,j} = \infty$ if i and j are positive but $S_{i,j}$ is not a square in T.

Definition 2.6. Suppose that $supp\, D(k, T) = \{S_{i,j}, S_{i+1,j}\}$ or $\{S_{i,j-1}, S_{i,j}\}$ and the square $S_{i,j}$ is fixed. Define $D'(k)$ to be a domino labeled by the integer k with $supp\, D'(k, T)$ equal to

(1) $\{S_{i,j}, S_{i-1,j}\}$ if $k < label\, S_{i-1,j+1}$
(2) $\{S_{i,j}, S_{i,j+1}\}$ if $k > label\, S_{i-1,j+1}$

Alternately, suppose that $supp\, D(k, T) = \{S_{i,j}, S_{i-1,j}\}$ or $\{S_{i,j+1}, S_{i,j}\}$ and the square $S_{i,j}$ is fixed. Define $supp\, D'(k, T)$ to be

(1) $\{S_{i,j}, S_{i,j-1}\}$ if $k < label\, S_{i+1,j-1}$
(2) $\{S_{i,j}, S_{i+1,j}\}$ if $k > label\, S_{i+1,j-1}$

Definition 2.7. The cycle $c = c(k, T)$ through k in a standard domino tableau T is a union of labels of T defined by the condition that $l \in c$ if either

(1) $l = k$,
(2) $supp\, D(l, T) \cap supp\, D'(m, T) \neq \emptyset$ for some $m \in c$, or
(3) $supp\, D'(l, T) \cap supp\, D(m, T) \neq \emptyset$ for some $m \in c$.

We will often identify the labels contained in the cycle with their underlying dominos. For a standard domino tableau T of rank r and a cycle c in T, we can define a domino tableau $MT(T, c)$ by replacing every domino $D(l, T) \in c$ by the corresponding domino $D'(l, T)$. That the resulting tableau $MT(T, c)$ is standard follows from [3], (1.5.27). In general, the shape of $MT(T, c)$ will either equal the shape of T, or one square will be removed (or added to the core) and one will be added. The cycle c is called *closed* in the former case and *open* in the latter. For an open cycle c of a tableau T, we will write $S_b(c, T)$ and $S_f(c, T)$ for the squares that have been removed (or added to the core) and added by moving through c; we will often abbreviate this notation to $S_b(c)$ and $S_f(c)$ when no confusion can

result. Let U be a set of cycles in T. According to [3], (1.5.29), the order in which one moves through a set of cycles does not matter, allowing us to unambiguously write $MT(T,U)$ for the tableau obtained by moving-through all of the cycles in U.

We next define the set of cycles that it will be necessary to move through to describe the relationship between G_r and G_{r+1}.

For $T \in SDT_r(n)$, we will write $\delta = \delta(T)$ for the set of squares $S_{i,j}$ that satisfy $i+j = r+2$. These are the squares with positive labels adjacent to the core of T. All are variable in our choice of fixed and variable squares. In order to obtain a domino tableau of rank $r+1$, it will be necessary to clear all of the squares in δ. Simply moving through $\Delta(T)$, the cycles in T that pass through δ, will achieve this effect. However, when applied to a pair of tableaux of the same shape, the resulting pair of tableaux may not be of the same shape. To this effect, we would like to define a minimal set of cycles in a pair of domino tableaux that will ensure this. More precisely, for a pair (T,S), we would like to find sets of cycles $\gamma = (\gamma(T), \gamma(S))$ in both T and S with $\Delta(T) \subset \gamma(T)$ and $\Delta(S) \subset \gamma(S)$ such that $MT(T, \gamma(T))$ and $MT(S, \gamma(S))$ have the same shape.

The natural notion to consider is an extended cycle ([4], (2.3.1)), which we now reconstruct.

Definition 2.8. Consider (T,S) a pair of same-shape domino tableaux, k a label of a domino in T, and c the cycle in T through k. The extended cycle $\tilde{c}$ of k in T relative to S is a union of cycles in T which contains c. Further, the union of two cycles $c_1 \cup c_2$ lies in $\tilde{c}$ if either is contained in $\tilde{c}$ and, for some cycle d in S, $S_b(d)$ coincides with a square of c_1 and $S_f(d)$ coincides with a square of $MT(T, c_2)$. The symmetric notion of an extended cycle in S relative to T is defined in the natural way.

Let $\tilde{c}$ be an extended cycle in T relative to S. According to the definition, it is possible to write $\tilde{c} = c_1 \cup \ldots \cup c_m$ and find cycles $d_1, \ldots, d_m$ in S such that $S_b(c_i) = S_b(d_i)$ for all i, $S_f(d_m) = S_f(c_1)$, and $S_f(d_i) = S_f(c_{i+1})$ for $1 \le i < m$. The union $\tilde{d} = d_1 \cup \cdots \cup d_m$ is an extended cycle in S relative to T called the *extended cycle corresponding to* $\tilde{c}$. Symmetrically, $\tilde{c}$ is the extended cycle corresponding to $\tilde{d}$.

It is now possible to define a moving through operation for a pair of same-shape domino tableaux. If we let b be the ordered pair $(\tilde{c}, \tilde{d})$ of extended cycles in (T,S) that correspond to each other, then we define

$$MT((T,S),b) = (MT(T,\tilde{c}), MT(S,\tilde{d})).$$

As desired, this operation produces another pair of same-shape domino tableaux ([4], (2.3.1)). If B is a family of ordered pairs of extended cycles that correspond to each other, then we can unambiguously define $MT((T,S),B)$, the operation of moving through all of the pairs simultaneously.

3. A Domino Tableau Correspondence

From the definitions of the previous section, it is apparent that moving through all of the extended cycles that pass thorough $\delta(T)$ and $\delta(S)$ of a same-shape domino tableau pair (T,S) will not only increase the rank of the resulting tableau pair by one, but the two tableaux will also be of the same shape. What is perhaps surprising is that this map, which merely evacuates δ in the simplest manner that will keep

the domino tableau pair of the same shape, describes the relationship between the Robinson-Schensted maps G_r and G_{r+1}.

3.1. **Main Theorem.** We first simplify our notation slightly. Consider a pair of domino tableaux (T,S) of rank r and define $\gamma(T)$ to be the set of extended cycles in T through $\delta(T)$ relative to S. Similarly, let $\gamma(S)$ be the set of extended cycles in S through $\delta(S)$ relative to T. If we write γ for the ordered pair of sets of extended cycles $(\gamma(T),\gamma(S))$, then let

$$MMT((T,S)) = MT((T,S),\gamma)$$

be the minimal moving through map that clears all of the squares in $\delta(T)$ and $\delta(S)$.

Theorem 3.1. *Consider an element $\sigma \in H_n$. The Robinson-Schensted maps G_r and G_{r+1} for rank r and $r+1$ domino tableaux are related by*

$$G_{r+1}(\sigma) = MMT(G_r(\sigma)).$$

The proof is a direct consequence of the following lemma; we show that domino insertion commutes with moving through the set of extended cycles which pass through the squares adjacent to the cores of a domino tableau pair. We note that the lemma is not true when more general sets of cycles are considered.

Lemma 3.2. *Consider $\sigma \in H_n$. Then*

$$MMT\big(\alpha_{k+1}(G_r^k(\sigma))\big) = \alpha_{k+1}\big(MMT(G_r^k(\sigma))\big)$$

When $r=0$, the result is reminiscent of [4], (2.3.2). We follow a similar approach and redefine the scope of a number of technical statements to cover the situations possible in the set of rank r standard domino tableaux when $r \geq 0$.

Example 3.3. Consider $\sigma = ((2,1,-1),(1,2,1))$ in H_2. If $(T,S) = G_0(\sigma)$, then

$$T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \qquad\qquad S = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

The cycles in T are $c_1 = \{1\}$ and $c_2 = \{2\}$ and the cycles in S are $d_1 = \{1\}$ and $d_2 = \{2\}$. Note that $\Delta(T) = c_1$ and $\Delta(S) = d_1$. However, $\gamma(T) = c_1 \cup c_2$ and $\gamma(S) = d_1 \cup d_2$, so that $MMT(G_0(\sigma))$ is the pair of tableaux

$$T' = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} \qquad\qquad S' = \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

As stated in the theorem, $MMT(G_0(\sigma)) \equiv (T',S')$ equals $G_1(\sigma)$.

3.2. **Technical Lemmas.** It is possible to describe the open cycles in $T(k+1)$ in terms of the open cycles in $T(k)$. Garfinkle's [4], (2.2.3) describes this relationship when $r=0$. With only minor changes, this result can be stated for arbitrary rank tableaux. We will write $OC(T)$ for the set of open cycles in T. To be precise, let us recall a definition:

Definition 3.4. If $T_1, T_2 \in SDT_r(n)$, and U_1 and U_2 are sets of open cycles in T_1 and T_2, then a map $\mu : U_1 \to U_2$ is a *cycle structure preserving bijection* if for every $c \in U_1$, $S_b(\mu(c)) = S_b(c)$ and $S_f(\mu(c)) = S_f(c)$.

In general, there is no cycle structure preserving bijection between the open cycles in $T(k+1)$ and those in $T(k)$. However, their relationship is only slightly more subtle.

Definition 3.5. A cycle $c \in OC(T(k+1))$ corresponds to a cycle $c' \in OC(T(k))$ if either $S_b(c') = S_b(c)$ or $S_f(c) = S_f(c')$.

We will describe the open cycle correspondences and cycle structure preserving bijections between $T(k+1)$ and $T(k)$. The first lemma is a generalized version of [4], (2.2.3), extended by the case here labeled as $2(a)(ii)$. Before stating it, let us introduce notation that will be used throughout this section. We will write T for $T(k+1)$, T' for $T(k)$, and $\overline{U}$ for the tableau U with its highest-labeled domino removed. Let P be the squares in T that are not in T' and $\overline{P}$ be the squares in $\overline{T}$ that are not in $\overline{T}'$. If e is the highest label in T, let P'_e be the squares of $D(e, T')$, and P_e be the squares of $D(e, T)$.

Lemma 3.6. *Consider $T(k)$ and $T(k+1) \in SDT_r(n)$. Suppose P is horizontal and consists of the squares $\{S_{ij}, S_{i,j+1}\}$. When P is vertical instead, the obvious transpositions of the below statements are true. The relationship of the open cycle structure of $T(k)$ to the open cycle structure of $T(k+1)$ is described by the following cases:*

(1) *Suppose $S_{i,j+1}$ is variable.*
 (a) *First assume that $j > 1$ and $S_{i+1,j-1}$ is not contained in the diagram underlying $T(k)$. Let c' be the open cycle in $T(k)$ with $S_b(c') = S_{i,j-1}$. Then there is an open cycle c in $T(k+1)$ with $S_f(c) = S_f(c')$ and $S_b(c) = S_{i,j+1}$. Furthermore, there is a cycle structure preserving bijection between the remaining open cycles of $T(k)$ and $T(k+1)$.*
 (b) *Otherwise, either $j = 1$ or $S_{i+1,j-1}$ is contained in the diagram underlying $T(k)$. Then there are two possibilities. Either*
  (i) *there is an open cycle c in $T(k+1)$ with $S_b(c) = S_{i,j+1}$ and $S_f(c) = S_{i+1,j}$ and a cycle structure preserving bijection between $OC(T(k))$ and $OC(T(k+1)) \setminus \{c\}$, or*
  (ii) *there is an open cycle c' in $T(k)$ and cycles c_1, c_2 in $T(k+1)$ such that $S_f(c_1) = S_f(c')$, $S_b(c_1) = S_{i,j+1}$, $S_f(c_2) = S_{i+1,j}$, and $S_b(c_2) = S_b(c')$. In this case, there is a cycle structure preserving bijection between $OC(T(k)) \setminus \{c'\}$ and $OC(T(k+1)) \setminus \{c_1, c_2\}$.*

(2) *Suppose $S_{i,j+1}$ is fixed.*
 (a) *First assume that either $i = 1$ or $S_{i-1,j+2}$ is contained in the diagram underlying $T(k+1)$. There are two possibilities. Either*
  (i) *there is an open cycle c' in $T(k)$ with $S_f(c') = S_{ij}$ and an open cycle c in $T(k+1)$ with $S_f(c) = S_{i,j+2}$ and $S_b(c) = S_b(c')$; in this case there is a cycle structure preserving bijection between the remaining open cycles of $T(k)$ and $T(k+1)$, or*
  (ii) *$S_{ij} \in \delta(T(k))$, there is a cycle c in $T(k+1)$ with $S_b(c) = S_{i,j}$ and $S_f(c) = S_{i,j+2}$, and a cycle structure preserving bijection between $OC(T(k))$ and $OC(T(k+1)) \setminus \{c\}$.*
 (b) *Otherwise, both $i > 1$ and $S_{i-1,j+2}$ is not contained in the diagram underlying $T(k+1)$. Then there is an integer $u > \sigma_{k+1}$ such that the domino with label u forms a cycle c' in $T(k)$ with $S_f(c') = S_{ij}$ and $S_b(c') = S_{i-1,j+1}$. In this case, there is a cycle structure preserving bijection between $OC(T(k)) \setminus \{c'\}$ and $OC(T(k+1))$.*

To verify the above, it is necessary to understand how the cycle structure of a domino tableau U is related to the cycle structure of $\overline{U}$. When $r = 0$, this is

described in [4], (2.2.4). Again for completeness, we state our version for arbitrary rank tableaux in full, which differs in the additional case $2(a)(ii)$. The proof of this lemma follows from an easy, but tedious, inspection.

Lemma 3.7. *Suppose that $T \in SDT_r(n)$, e is the label of its highest domino D, and $\overline{T}$ is the domino tableau with D removed. Suppose D occupies the squares $\{S_{ij}, S_{i,j+1}\}$ in T. Again, the obvious transpositions of the statements below are true for vertical D.*

- *(1) Suppose that $S_{i,j+1}$ is variable.*
  - *(a) First assume that $j > 1$ and $S_{i+1,j-1}$ is not contained in the diagram underlying $\overline{T}$. Let $\overline{c}$ be the open cycle in $\overline{T}$ with $S_b(\overline{c}) = S_{i,j-1}$. Then there is an open cycle c in T with $S_f(c) = S_f(\overline{c})$ and $S_b(c) = S_{i,j+1}$. Furthermore, there is a cycle structure preserving bijection between the remaining open cycles of $\overline{T}$ and T.*
  - *(b) Otherwise, either $j = 1$ or $S_{i+1,j-1}$ is contained in the diagram underlying $\overline{T}$. Then $c = \{e\}$ is an open cycle in T and there is a cycle structure preserving bijection between $OC(\overline{T})$ and $OC(T) \setminus \{c\}$.*
- *(2) Suppose that $S_{i,j+1}$ is fixed.*
  - *(a) First assume that either $i = 1$ or $S_{i-1,j+2}$ is contained in the diagram underlying T.Then there are two possibilities. Either*
    - *(i) there exists an open cycle $\overline{c}$ in $\overline{T}$ with $S_f(\overline{c}) = S_{ij}$, and $c = \overline{c} \cup \{e\}$ is an open cycle in T; in this case there is a cycle structure preserving bijection between $OC(\overline{T}) \setminus \{\overline{c}\}$ and $OC(T) \setminus \{c\}$, or*
    - *(ii) $S_{ij} \in \delta(T)$, there is a cycle c in T with $S_b(c) = S_{ij}$ and $S_f(c) = S_{i,j+2}$, and a cycle structure preserving bijection between $OC(\overline{T})$ and $OC(T) \setminus \{c\}$.*
  - *(b) Otherwise, both $i > 1$ and $S_{i-1,j+2}$ is not contained in the diagram underlying T. Then either*
    - *(i) there is a cycle $\overline{c}$ in $\overline{T}$ with $S_b(\overline{c}) = S_{i-1,j+1}$ and $S_f(\overline{c}) = S_{ij}$, $c = \overline{c} \cup \{e\}$ is a closed cycle in T, and $OC(T) = OC(\overline{T}) \setminus \{\overline{c}\}$, or*
    - *(ii) there are two open cycles $\overline{c}_1, \overline{c}_2$ in $\overline{T}$ such that $S_b(\overline{c}_1) = S_{i-1,j+1}$, $S_f(\overline{c}_2) = S_{ij}$, the set $c = c_1 \cup c_2 \cup \{e\}$ is an open cycle in T and $OC(T) \setminus \{c\} = OC(\overline{T}) \setminus \{\overline{c}_1, \overline{c}_2\}$.*

Armed with this observation, we can now prove Lemma 3.6.

Proof. Lemma 3.7 describes the relationships between the cycle structures of $\overline{T}(k)$ and $T(k)$, as well as $\overline{T}(k+1)$ and $T(k+1)$. If we use induction on the size of the tableaux, we can relate the cycle structures of $\overline{T}(k)$ and $\overline{T}(k+1)$. Together, this allows us to describe the desired relationship between the cycle structures of $T(k)$ and $T(k+1)$.

If a pair of squares in a domino tableau satisfy the hypotheses of a case of Lemma 3.6 or Lemma 3.7, we will say that the pair lies in the situation labeled by that case. The proof of the lemma divides into different cases described by the situations of $\overline{P}$ and P'_e and their relative positions. When $r = 0$, this is exhaustively carried out in the proof of [4], (2.2.3), which includes a description of the possibilities for $\overline{P}$ and P'_e. We will use the same labels for these possibilities. To verify the lemma for arbitrary rank tableaux, we must check that the conclusions still hold in the cases originally considered, as well as examine the new cases that arise for larger rank

tableaux. The former follows from a lengthy inspection of the proof of [4], (2.2.3). We examine the new cases.

We have to consider situations where either $P, \overline{P}, P'_e$, or P_e is in situation $2(a)(ii)$. Most of the cases are essentially trivial. We treat two of them in detail; the rest follow along similar lines. The cases are labeled to mimic similar cases considered in [4], (2.2.3).

Case K′. Here $\overline{P} = P'_e$ is in situation $2(a)(ii)$. We have a cycle structure preserving bijection between $OC(\overline{T})$ and $OC(T')$. Note that $P_e = P$, and they both must be in situation $2(a)(ii)$ or $1(b)$. In both cases, the desired relationship between $OC(T)$ and $OC(T')$ exists between $OC(T)$ and $OC(\overline{T})$ by Lemma 3.7. Since we already have a cycle structure preserving bijection between $OC(\overline{T})$ and $OC(T')$, we are done.

Case L′. Here $\overline{P}$ is in situation $2(a)(ii)$ and $P'_e = \{S_{i,j}, S_{i+1,j}\}$, so that P'_e is in situation $2(a)(ii)$ as well. If D is the domino in $\overline{T}$ in position $\overline{P}$ with label f, then we have a cycle structure preserving bijection between $OC(\overline{T}) \setminus \{f\}$ and $OC(\overline{T}') \setminus \{e\}$. Note that $P = \{S_{i,j+1}, S_{i+1,j+1}\}$ is in situation $1(b)$ of 3.6 and $P_e = \{S_{i+1,j}, S_{i+1,j+1}\}$ is in situation $1(b)$ of Lemma 3.7. Because of the latter, we know there is a cycle structure preserving bijection between $OC(\overline{T})$ and $OC(T) \setminus \{e\}$. From this, we can construct a cycle structure preserving bijection between $OC(T') \setminus \{c'\}$ and $OC(T) \setminus \{c_1, c_2\}$ where $c' = \{e\}$ in T', $c_1 = \{e\}$ in T, and $c_2 = \{f\}$, as required in the conclusion of $1(b)(ii)$.

□

Lemma 3.8. *The set $\gamma(T(k+1))$ is the union of the open cycles that correspond to cycles in $\gamma(T(k))$ and the cycles through $\delta(T(k+1))$.*

Proof. Let us write $\tilde{\gamma}(T')$ for the set of open cycles in T that correspond to open cycles in $\gamma(T')$. We may take $k > 1$, otherwise this is trivial. First assume that $\sigma_{k+1} = e$. Then $P_e = \{S_{1,s}, S_{1,s+1}\}$ and could be in situations $1(a)$, $1(b)$, $2(a)(i)$, or $2(a)(ii)$ of Lemma 3.7. In the first and third cases, let c' be the cycle in T' through the square $S_{1,s-1}$. Then $c = c' \cup \{e\}$ is the open cycle in T corresponding to c', $OC(T') \setminus \{c'\} = OC(T) \setminus \{c\}$, and $c \in \gamma(T)$ iff $c' \in \gamma(T')$. Since $\Delta(T) \subset \tilde{\gamma}(T')$, the result follows. If P_e is in situation $1(b)$ of Lemma 3.7, then $OC(T') = OC(T) \setminus \{e\}$. Since $\{k+1\}$ is a cycle in S, $\{e\}$ must be an extended cycle implying that $\{e\} \notin \gamma(T)$. Again, $\Delta(T) \subset \tilde{\gamma}(T')$ and the result follows. If P_e is in situation $2(a)(ii)$ of Lemma 3.7, then $\{k+1\}$ is a cycle in S, $\{e\}$ must be an extended cycle and since $\{e\} \in \Delta(T)$, the result follows.

The rest of the proof is by induction on the size of the tableau. We will assume that $\gamma(\overline{T}) = \tilde{\gamma}(\overline{T}') \cup \Delta(T)$. We treat cases A-C and L from the proof of [4], (2.2.3) incorporating the additional possibilities that arise in higher rank tableaux. Remaining cases are handled along similar lines.

Case A. Suppose $\overline{P}$ is in situation $1(a)$ and $\overline{P} = P'_e$. Then $P = P_e$ and they both equal to the set $\{S_{i+1,s}, S_{i+1,s+1}\}$ for some s. The squares of P may be in situations $1(a)$, $1(b)$, $2(a)(i)$, or $2(a)(ii)$ of Lemma 3.6. In the first case, consider c' as in Lemma 3.6($1(a)$). The cycle c' corresponds to $c = c(e, T)$ since $S_f(c) = S_f(c')$. Examining the position of $D(k+1, S)$, we find that the rest of the extended cycle structure of T is the same as in T'. Hence if c is any cycle in T that corresponds to a cycle c' in T', then $c \in \gamma(T)$ iff $c' \in \gamma(T')$. If P lies in situation $2(a)(ii)$, then

$S_{i+1,s} \in \delta(T)$, $c(e,T)$ is a cycle through $\delta(T)$ and lies in $\gamma(T)$. Similar arguments work for the remaining two cases.

Case B. Here $\overline{P}$ is in situation $1(a)$ and $P'_e = \{S_{i+1,j-1}, S_{i+2,j-1}\}$, implying that $P = \overline{P}$ and $P_e = P'_e$. First consider the cycle $c = c(e,T) = \{e\}$. Note that c corresponds to $c' = c(e,T')$ since $S_b(c) = S_b(c')$. Let $\overline{c}' = c' \setminus \{e\} \in OC(\overline{T}')$. Let $f = label(S_{i,j+1}, T)$ and note that the squares of P form a domino in S with label $k+1$. Then $S_b(k+1,S) = S_b(f,T)$ and $S_f(k+1,S) = S_f(e,T)$, so that e and f are both in the same extended cycle of T relative to S. Hence $e \in \gamma(T)$ iff $f \in \gamma(T)$ iff $f \in \gamma(\overline{T})$ iff $\overline{c}' \in \gamma(T')$ iff $c' \in \gamma(T')$, as desired. For any open cycle c not containing e in T, the result follows by induction.

Case C. Here $\overline{P}$ is in situation $1(a)$ and $P'_e = \{S_{i+1,j-2}, S_{i+1,j-1}\}$ is in situation $2(b)(i)$. Then $P = \overline{P}$ and $P_e = P'_e$. Let $c = c(e,T)$ and by the conclusion of Lemma 3.7 we find $S_f(c) = S_{i+1,j}$ and $S_b(c) = S_{i,j+1}$. Note that c corresponds to no open cycles in T'. Since $S_f(c,T) = S_f(k+1,S)$ and $S_b(c,T) = S_b(k+1,S)$, the extended cycle of e is just c. Hence $c \in \gamma(T)$ iff c passes through $\delta(T)$. For any open cycle c not containing e in T, the result follows by induction.

Case L. Consider $\overline{P}$ in situation $2(a)(ii)$ and $P'_e = \{S_{i,j}, S_{i+1,j}\}$, so that P'_e is in situation $2(a)(ii)$ as well. We then have $P = \{S_{i,j+1}, S_{i+1,j+1}\}$ and $P_e = \{S_{i+1,j}, S_{i+1,j+1}\}$. First consider the cycle $c = c(e,T) = \{e\}$. Note that $\overline{P}$ is a domino in T, say with label f, and P'_e is a domino in S, say with label l. Then $S_b(c(l,S),S) = S_{i,j} = S_b(c(f,T),T)$ and $S_f(c(l,S),S) = S_{i+2,j} = S_f(c(e,T),T)$. Hence $\{e\}$ lies in the extended cycle through $c(f,T)$. Since $c(f,T) \in \Delta(T)$, we must have $\{e\} \in \gamma(T)$. If we let $c' = c(e,T')$, then $S_f(c) = S_f(c')$, which means that c corresponds to c'. In other words, $\{e\}$ lies in $\gamma(T)$ and $\tilde{\gamma}(T') \cup \Delta(T)$. Finally, consider any open cycle c not containing e in T. Then c is also an open cycle in $\overline{T}$, and the rest follows by induction. We omit the argument when $\overline{P}$ and P'_e are in situation $2(a)(i)$ instead.

$\square$

If we abuse notation and write $MMT(T)$ for $MT(T, \gamma(T))$, then we can state the following version of Garfinkle's [4], (2.2.9), which verifies Lemma 3.2 for left tableaux.

Lemma 3.9. *Consider $\sigma \in H_n$ and write $T(m)$ for the left tableau of $G_r^m(\sigma)$. Then*

$$\alpha_{k+1}\big(MMT(T(k))\big) = MMT\big(T(k+1)\big).$$

Proof. Using Lemma 3.8, we have to show that

$$\alpha_{k+1}\big(MMT(T(k))\big) = MT(T(k+1), \tilde{\gamma}(T(k)) \cup \Delta(T(k+1))).$$

which is an adaptation of [4], (2.2.9). However, we cannot adapt the proof of [4], (2.2.9) verbatim, as it uses induction on the number of open cycles in the extended cycle defining the moving through operation. In our situation, moving through a set of cycles smaller than $\gamma(T(k))$ may leave us with a domino tableau on which α is undefined. Nevertheless, since only one pair P of squares is added to $T(k)$ with domino insertion, and moving through open cycles can be done independently, we can essentially follow the original proof and examine the relationship of P with the cycles in $\gamma(T(k))$ individually.

The case when $\sigma_{k+1} = e$ is simple, and we assume that $\sigma_{k+1} \neq e$. We proceed by induction on n, noting that the case $n = 1$ corresponds to $\sigma_{k+1} = e$. Following the original proof of [4], (2.2.9), we show that each domino in $\alpha_{k+1}\big(MMT(T')\big)$

lies in the same position in $MMT(T)$. For dominos with labels less than e, this will follow by induction; for the domino with label e, it will follow by inspection of each of the cases below.

Let $\overline{P}_1$ be the squares in $\alpha_{k+1}(\overline{T}')$ that are not in $\overline{T}'$, $\overline{P}_2$ be the squares in $\alpha_{k+1}(MMT(\overline{T}'))$ that are not in $MMT(\overline{T}')$. Write T_1 for T, T_1' for T', T_2 for $MMT(T)$, T_2' for $MMT(T')$, and T_3 for $\alpha_{k+1}(MMT(T'))$. Hence we are verifying that $T_2 = T_3$.

Case A. Assume that $\overline{P}_1 = P_e' = \{S_{ij}, S_{i,j+1}\}$, and $\overline{P}_1$ is in situation $1(a)$. Then $P_e = P = \{S_{i+1,s}, S_{i+1,s+1}\}$ for some s. Suppose first that $S_{i+1,s}$ is variable and that no cycle $c' \in \gamma(T_1')$ has $S_f(c') = S_{i+1,s}$. If $S_{i+1,s} \in \delta$, then $\{e\} \in \gamma(T_1)$ and $P_e(T_2) = \{S_{i+1,s+1}, S_{i+1,s+2}\} = P_e(T_3)$. When $S_{i+1,s} \notin \delta$, we have $P_e(T_2) = P_e(T_1) = P_e(T_3)$. Suppose next that there is a cycle $c' \in \gamma(T_1')$ with $S_f(c') = S_{i+1,s}$, then e lies in a cycle in $\gamma(T_1)$ and $P_e(T_2) = \{S_{i+1,s+1}, S_{i+1,s+2}\} = P_e(T_3)$. If $S_{i+1,s}$ is fixed, then $P_e(T_2) = \{S_{i+1,s-1}, S_{i+1,s}\} = P_e(T_3)$ if $S_{i+1,s-1}$ lies in some cycle of $\gamma(T_1')$, and $P_e(T_2) = \{S_{i+1,s}, S_{i+1,s+1}\} = P_e(T_3)$ if it does not.

Case K$'$. Here $\overline{P}_1 = P_e'$ are in situation $2(a)(ii)$. Then $P_e = \{S_{i+1,j}, S_{i+1,j+1}\}$. Note that $c = \{e\}$ is a cycle in T_1 and $d = \{k+1\}$ is a cycle in $S(k+1)$ with $S_f(c, T_1) = S_{i+2,j} = S_f(d, S(k+1))$ and $S_b(c, T_1) = S_{i+1,j+1} = S_b(d, S(k+1))$. Hence $c = \{e\}$ is an extended cycle not contained in $\gamma(T_1)$ and consequently $D(e, T_2) = D(e, T_1) = \{S_{i+1,j}, S_{i+1,j+1}\}$. Now note that $\overline{P}_2 = P(e, T_2')$ and by a similar argument, we obtain $D(e, T_3) = \{S_{i+1,j}, S_{i+1,j+1}\}$, as desired.

Case L$'$. Here $\overline{P}_1$ is in situation $2(a)(ii)$ and $P_e' = \{S_{ij}, S_{i+1,j}\}$, so it is in situation in $2(a)(ii)$ as well. Then $P_e = \{S_{i+1,j}, S_{i+1,j+1}\}$ and $P = \{S_{i,j+1}, S_{i+1,j+1}\}$. Note that $c' = \{e\}$ is a cycle in T_1' with $S_f(c) = S_{i+2,j}$ and that the squares P_e' form a domino in $S(k)$, say with label f. Let $d = c(f, S(k+1))$ and note $d \in \gamma(S(k+1))$. Furthermore, $S_f(d) = S_{i+2,j}$ implying that $c(e, T_1) \in \gamma(T_1)$, and $D(e, T_2) = \{S_{i+1,j}, S_{i+2,j}\}$. Now observe that $D(e, T_2') = \{S_{i+1,j}, S_{i+2,j}\}$ and $\overline{P}_2 = \{S_{i,j+1}, S_{i,j+2}\}$. This means $D(e, T_3) = \{S_{i+1,j}, S_{i+2,j}\}$, and $D(e, T_2) = D(e, T_3)$, as desired.

□

3.3. **Domino Insertion and Moving Through.** Armed with the technical results of the previous section, we can now address the main lemma of the paper. We prove Lemma 3.2, verifying that domino insertion on tableau pairs commutes with the minimal moving through map. Write $(T_1', S_1') = (T(k), S(k))$, $(T_1, S_1) = (T(k+1), S(k+1))$, $(T_2', S_2') = MMT(T_1', S_1')$, $(T_2, S_2) = \alpha_{k+1}((T_2', S_2'))$, and $(T_3, S_3) = MMT(T_1, S_1)$. Expressed in this notation, we would like to prove that $(T_2, S_2) = (T_3, S_3)$. Lemma 3.9 says that $T_2 = T_3$, and it remains to show that $S_2 = S_3$.

Proof. Write P_1 for the squares in T_1 that are not in T_1' and P_2 for the squares in T_2 that are not in T_2'. Note that P_1 forms a domino in S_1 and P_2 forms a domino in S_2, both with label $k+1$. Assume that $P_1 = \{S_{i,j}, S_{i,j+1}\}$. We will examine the cases when P_1 is in situations $1(a)$, $1(b)(ii)$, and $2(b)$. The others follow along similar lines.

So suppose that P_1 is in situation $1(a)$ of Lemma 3.6 and c is the open cycle with $S_b(c) = S_{i,j+1}$ described therein. Then $D(k+1, S_1)$ is in situation $1(a)$ of Lemma 3.7 and there is an open cycle $\overline{d}$ in S_1' with $S_b(\overline{d}) = S_{i,j-1}$ such that $d = \overline{d} \cup \{k+1\}$ is an open cycle in S_1. Note that $c \in \gamma(T_1)$ iff $d \in \gamma(S_1)$. If $c \in \gamma(T_1)$, then by

Lemma 3.9, $P_2 = \{S_{i,j-1}, S_{ij}\}$, which implies that $D(k+1, S_3) = D(k+1, S_2)$. Since the rest of the cycle structure in T_1 remains the same as in T_1', the rest of the cycles in $\gamma(S_1)$ are the same as in $\gamma(S_1')$ and consequently, $S_2 = S_3$. If $c \notin \gamma(T_1)$, the result is clear.

If P_1 is in situation $1(b)(ii)$ of Lemma 3.6, then $D(k+1, S_1)$ is in situation $1(b)$ of Lemma 3.7. Let c', c_1, and c_2 be as described in Lemma 3.6 $1(b)(ii)$ and let $d = c(k+1, S_1)$. Since $S_b(c_1) = S_b(d)$ and $S_f(c_2) = S_f(d)$, c_1 and c_2 lie in the same extended cycle relative to d, so $c_1, c_2 \in \gamma(T_1)$ iff $d \in \gamma(S_1)$. If $c_1, c_2 \in \gamma(T_1)$, then by Lemma 3.9, $P_2 = \{S_{ij}, S_{i+1,j}\}$. Since $d \in \gamma(S_1)$, this means $D(k+1, S_2) = D(k+1, S_3)$. Since the rest of the cycle structure in T_1 remains the same as in T_1', the rest of the cycles in $\gamma(S_1)$ are the same as in $\gamma(S_1')$ and we can conclude that $S_2 = S_3$. If $c_1, c_2 \notin \gamma(T_1)$, the conclusion is the same.

The most troublesome case is when P_1 is in situation $2(b)$ of Lemma 3.6. Then $D(k+1, S_1)$ is either in situation $2(b)(i)$ or $2(b)(ii)$ of Lemma 3.7. So suppose first that $D(k+1, S_1)$ is in situation $2(b)(i)$. Let $\overline{d}$ be the cycle in S_1' with $S_f(\overline{d}) = S_{i,j}$ and $S_b(\overline{d}) = S_{i-1,j+1}$. Then $d = \overline{d} \cup \{k+1\}$ is a closed cycle in S_1 and consequently does not lie in $\gamma(S_1)$. Let c' be the cycle in T_1' with $S_f(c') = S_{ij}$ and $S_b(c') = S_{i-1,j+1}$. Then c' is the entire extended cycle in T_1' that corresponds to $\overline{d}$ in S_1'; in particular, this means that $c' \notin \gamma(T_1')$ and $\overline{d} \notin \gamma(S_1')$. Consequently, $S_2 = S_3$.

Finally, consider $D(k+1, S_1)$ in situation $2(b)(ii)$. Let d_1 and d_2 be the cycles in S_1' with $S_b(d_1) = S_{i-1,j+1}$ and $S_f(d_2) = S_{ij}$. Then $d_1 \cup d_2 \cup \{k+1\}$ is an open cycle in S_1. Let c' be as in Lemma 3.6 2(b) and note that $c' \in \gamma(T_1')$ iff $d_1, d_2 \in \gamma(S_1')$. If $c' \in \gamma(T_1')$, then $P_2 = \{S_{i-1,j+1}, S_{i,j+1}\}$ by Lemma 3.9 and we again conclude that $S_2 = S_3$. If $c' \notin \gamma(T_1')$, the result is clear.

$\square$

3.4. **Restriction to Involutions.** We follow van Leeuwen in the next definition, which constructs a map between domino tableaux of unequal rank [11].

Definition 3.10. Let r and r' be non-negative integers and suppose that $T \in SDT_r(n)$. We define the map $t_{r,r'} : SDT_r(n) \to SDT_{r'}(n)$ by setting $t_{r,r'}(T) = T'$ whenever $G_r^{-1}(T,T) = G_{r'}^{-1}(T', T')$.

Armed with Theorem 3.1, the maps $t_{r,r+1}$ take a particularly simple form. The domino tableau $t_{r,r+1}(T)$ in $SDT_{r+1}(n)$ is simply the image of T after all the cycles in $\Delta(T)$ have been moved through.

Corollary 3.11. $t_{r,r+1}(T) = MT(T, \Delta(T))$

Proof. If σ is an involution and $G_r(\sigma) = (T, S)$, then S must equal T. The definition of extended cycles implies that every extended cycle in T relative to S consists of a unique cycle. In our setting, this implies $\gamma = (\Delta(T), \Delta(T))$. Using Theorem 3.1 and the definition of moving through extended cycles, we now have that

$$\big(t_{r,r+1}(T), t_{r,r+1}(T)\big) = MT((T,T), \gamma) = \big(MT(T, \Delta(T)), MT(T, \Delta(T))\big),$$

as desired. $\square$

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